Fractals, Vol. 1, No. 4 (1993) 1030-1036.

SOC in a Class of Sandpile Models with Stochastic Dynamics

S. LÜBECK and K. D. USADEL

Theoretische Physik and SFB 166

Universität Duisburg, Lotharstraße 1, 47048 Duisburg, FRG

Abstract

We have studied one-dimensional cellular automata with updating rules depending stochastically on the difference of the heights of neighbouring cells. The probability for toppling depends on a parameter λ which goes to one with increasing slope, i.e. the dynamics can be varied continuously. We have investigated the scaling properties of the model using finite-size scaling analysis. A robust power-law behavior is observed for the probability density of the size of avalanches in a certain range of λ values. The two exponents which determine the dependence of the probability density on time and system size both depend continuously on λ, i.e. the model exhibits nonuniversal behavior. We also measured the roughness of the surface of the sandpile and here we obtained an universal behavior, i.e. a roughness exponent of about 1.75 for all values of λ. For the temporal behavior of the mass a $f^{-\phi}$ spectrum is obtained with an exponent ϕ close to 2 again for all λ-values.

1. INTRODUCTION

Bak, Tang and Wiesenfeld [1] recently introduced the notion of self-organized criticality (SOC) as a framework to understand the dynamics of extended dissipative dynamical systems. The dynamics drive the system into a state where relaxation processes occur over all time and length scales. It was suggested [1] that sandpiles were a particularly clear example of a self-organized system.

Theoretical sandpile models have been investigated by several authors [1]-[6]. They usually involve two kinds of steps. First a particle is added at ramdomly choosen sites, then the particles are eventually distributed to neighboring sites according to dynamical rules. Usually, these rules are deterministic: there is toppling of particles or not

depending on the local occupation of sites. However, real sandpiles behave in a different fashion. In real sandpiles the particles have different shapes, various sizes and they can stick together. As a matter of fact, large and very unstable slopes can develop with slides of particles in an apparently stochastic fashion: a certain slope may or may not lead to a slide depending on a great number of internal degrees of freedom like the above mentioned different shapes and sizes of the particles, the friction between them, the local variations in packing etc. To incorporate some of these features into a theoretical model we propose as a first step a model with dynamical rules which are not deterministic: toppling of particles occurs with a probability function which is small if the slope to neighboring sites is small and which increases to one with increasing slope. In the present paper we report results of extensive simulations of this model. We restrict ourselves to the one-dimensional case which shows already an interesting and nontrivial behavior.

2. THE MODEL AND SIMULATIONS

The model we consider is an one-dimensional "nonlocal unlimited" sandpile model [2] with the following rules. We assume integer heights h_i at lattice sites $i = 1, 2, 3, ..., L$, where L is the size of the system. The local slope σ_i at the site i is defined as the height difference between two neighbouring cells, $\sigma_i = h_i - h_{i+1}$. The dynamics obey the following rules: If the slope σ_i exceeds a threshold value σ_c an avalanche can be generated. With a certain probability $p(\sigma_i)$ grains drop from site i to the neighbouring cell with

$$h_i \to h_i - n_i$$

$$h_{i+j} \to h_{i+j} + 1 \qquad \text{for} \quad j = 1, 2, ..., n_i \tag{1}$$

The number n_i of grains which topple grows with increasing σ_i, $n_i = \sigma_i - N$ (unlimited model). Since there is toppling to n_i right neighbours the model is called nonlocal. Note that from Eq.(1) it follows that all slopes in the system are non-negative.

The simulation starts by adding particles at the top until the stability condition $\sigma_1 < \sigma_c$ is violated. Then, with probability $p(\sigma_1)$, n_1 grains flow to the right neighbour according to Eq.(1) and an avalanche develops. If no toppling takes place another grain is added to site 1. If toppling takes place the next site will be visited and so on. An avalanche stops at site i if this site doesn't topple or if its slope to site $i+1$ is too small. Thus an avalanche only starts at the top. The length of an avalanche is defined as the number of sliding sites. At the right edge of the system grains can leave the pile, i.e. h_i is set equal to zero for $i > L$.

The most important new ingredients in our model is the probability function $p(\sigma_i)$ which determines whether a certain site i topples or not. This function should obey the following conditions:

$$p(\sigma < \sigma_c) \equiv 0$$

$$p(\sigma') > p(\sigma) \qquad \text{for} \quad \sigma' > \sigma \tag{2}$$

$$p(\sigma) \to 1 \qquad \text{for} \quad \sigma \to \infty$$

The first condition states that there is only toppling if a threshold value is reached, the second conditions means that there is increasing probability for toppling if the slopes increase and the third equation means that toppling always takes place for very large slopes, i.e. the heights h_i are finite for any system of finite length. In our simulation we use

$$p(\sigma_i) = 1 - e^{-\lambda(\sigma_i - 1)} \tag{3}$$

λ is called the toppling parameter. Thus there are three parameters in our model: σ_c which determines the stability condition, N which determines the number of sliding grains in one event and most importantly the toppling parameter λ with which we can tune the dynamics continuously. In our simulations we choose $\sigma_c = 2$, $N = 1$ and study in particular the dependence of the model on λ. Note that for $\lambda \to \infty$ one gets a trivial model: $\sigma_i = 1$ for all i and any avalanche reaches the edge of the pile.
The most important quantity to study is the probability density for the lengths of the avalanches for which a finite-size scaling analysis is performed. As usual we assume for this probability density $P(t, L)$ a scaling form

$$P(t, L) = L^{-\beta} g(L^{-\nu} t) \tag{4}$$

where t is the number of toppling sites. For $t \ll L$ we expect a power-law behavior, $P(t, L) \sim t^{-\kappa}$ independent of L. Thus β,ν and κ must obey the following scaling relation [2]

$$\beta = \nu\kappa \tag{5}$$

Another important quantity is the roughness exponent ζ which a is measure for the fluctuations about the average height profile and which reflects the critical nature of the steady state. It is defined by [3]:

$$\xi = \sqrt{L^{-1} \sum_{i=1}^{L} < (h_i - < h_i >)^2 >} \sim L^{\zeta} \tag{6}$$

Finally we measure the total mass of the pile

$$M(t) = \sum_{i=1}^{L} h_i(t) \tag{7}$$

and subsequently calculated its Fourier spectrum $|M(f)|^2$ to determine the exponent ϕ.

3. RESULTS

The simulations were performed for lattice sizes of $L = 50, 100, 200, 500, 1000$. Before performing a measurement a large number of particles (arround 10^8) was added starting from an empty lattice to reach the critical state. Note that all measured values are independent of the initial conditions, i.e. the critical state is an attractor of the dynamics.
First we measured the probability density of the length of the avalanches $P(t, L)$ for a lattice size $L = 100$ and examined how $P(t, L)$ depends on the toppling parameter

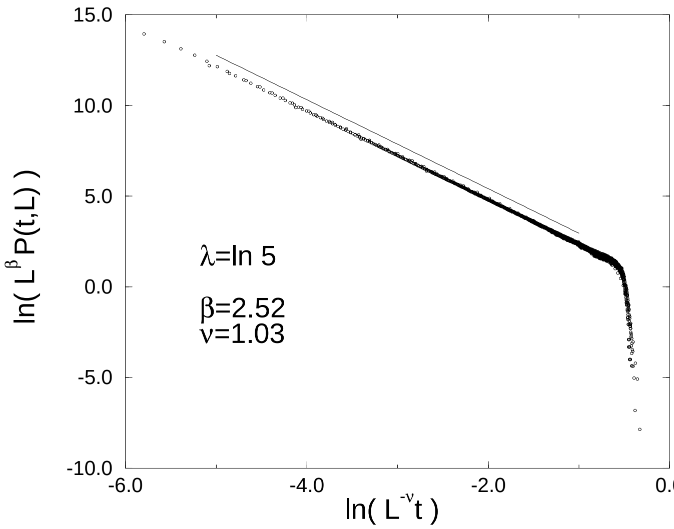

Figure 1: Finite-size scaling fit for $\lambda = ln\,5$ and five different lattice sizes $L = 50, 100, 200, 500, 1000$. The solid line corresponds to a power-law with an exponent $\kappa = 2.453$.

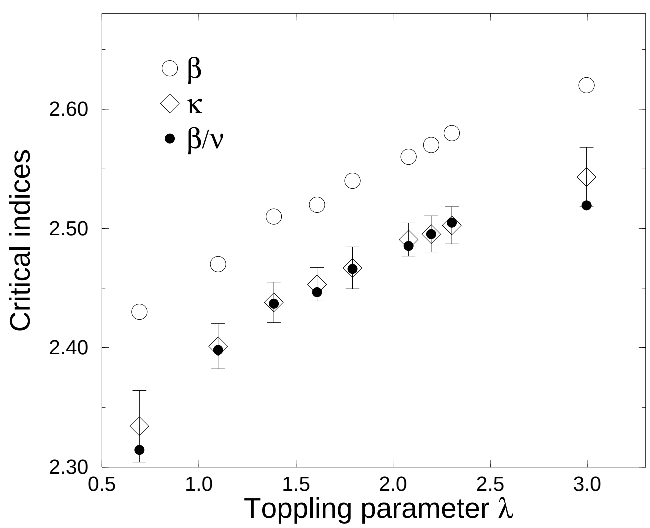

Figure 2: Scaling exponent β and power-law exponent κ as a function of the toppling parameter λ. Solid circles correspond to $\nu^{-1}\beta$. Eq. (5) is fulfilled within numerical accuracy.

λ. A power-law behavior is observed in the range $1 < \lambda < 3$. Outside this range we obtained on a log-log plot curves which deviate visible from a power-law behavior. We have not investigated this problem any further but restricted ourselves to the λ-values where power-law behavior is clearly observed. Then the probability density $P(t, L)$ was measured for various values of λ and lattice sizes. As can be seen from Fig.1 finite-size scaling works extremely well. For different values of λ we get $\nu \approx 1$ but the scaling exponent β changes strongly [Fig.2]. Thus we expect a varying exponent κ. κ is calculated using regression analysis for lattice sizes $L = 100, 200, 500, 1000$. The averaged value of κ is shown in Fig.2. Note that the change of the exponent can not be explained by statistical errors since they are far too small. The relation between the exponents, Eq.(5), is always fulfilled.

To determine the roughness exponents ζ we limit our simulation to lattice sizes $L \leq 500$. After reaching the critical state we add around 10^8 particles to calculate ξ for different

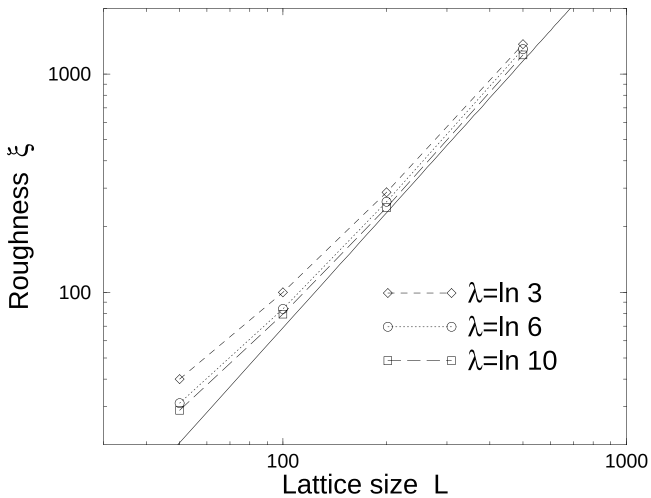

Figure 3: Roughness ξ as a function of lattice size for different values of the toppling parameter λ. The solid line corresponds to an power-law $\xi \sim L^{\zeta}$ with $\zeta = 1.75$.

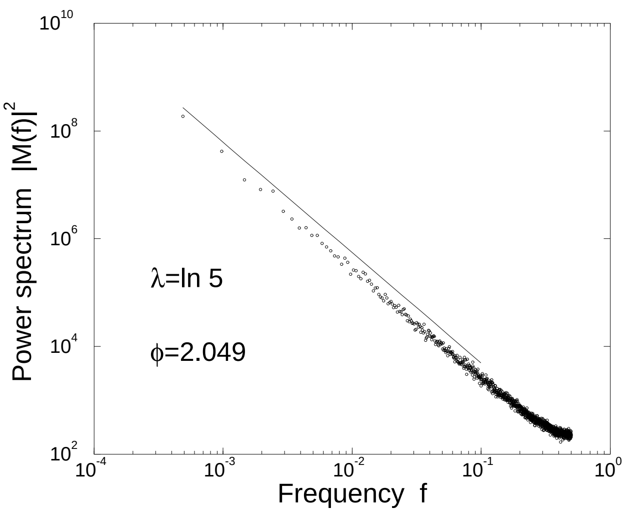

Figure 4: Power spectrum of the time evolution of the total mass of the pile for $\lambda = ln\,5$. The spectrum is an average over 40 measurements with 2048 events. Solid line corresponds to $|M(f)|^2 \sim f^{-2.049}$.

values of λ. Fig.3 shows the results. One can see that for $L \geq 100$ the roughness ξ scales with the system size as L^{ζ} and ζ is independent of the toppling parameter λ. However, one should note that for a very accurate determination of ζ one needs simulations with system sizes larger than 500 since the data points shown in Fig.3 are still not on a straight line. But those measurements would demand more than 10^9 events and have not yet been done. Increasing the lattice size the roughness grows very fast, because $\zeta \geq 1$. The average height profile scales as $< h_i > \sim L^{\alpha_1}$ with $\alpha_1 \approx 1.8$ for all i and λ [1]. The exponents α_1 and ζ should obey the relation $\alpha_1 \geq \zeta$ which is fulfilled in our model. It is very interesting to see whether the roughness exponent ζ is equal to the scaling exponent of the averaged height α_1. Finally we have calculated the time depen-

[1] Notice that the average height profile is not a linaer function of i, but obeys the scaling relation $< h_i >= L^{\alpha_1} f(L^{-\alpha_2} i)$ with $f = a_0 + a_1 e^{-a_2 i^2}$ [7].

Table 1: The critical exponents for different values of the toppling parameter λ.

λ	β	ν	κ	ζ	ϕ
$ln\,2$	2.43	1.03	2.334	1.59	2.048
$ln\,3$	2.47	1.03	2.401	1.70	2.061
$ln\,4$	2.51	1.03	2.438	1.75	2.051
$ln\,5$	2.52	1.03	2.453	1.80	2.049
$ln\,6$	2.54	1.03	2.467	1.76	2.053
$ln\,8$	2.56	1.03	2.491	1.75	2.064
$ln\,9$	2.57	1.03	2.495	1.77	2.058
$ln\,10$	2.58	1.04	2.503	1.76	2.112
$ln\,20$	2.62	1.05	2.543	1.75	2.030

dence of the total mass of the pile (Eq.7). We used a lattice size $L = 50$ and measured the mass $M(t)$ for 2048 events. Fig.4 shows the average over 40 of the corresponding fourier-spectra. The power spectrum $|M(f)|^2$ scales as $f^{-\phi}$ with $\phi \approx 2$ for all λ.

4. CONCLUSIONS

In conclusion we have simulated an one-dimensional cellular automaton with nonlocal, unlimited and stochastic dynamics which exhibits SOC behavior in a certain range of λ-values. All trademarks of SOC are observed: finite-size scaling and a power-law behavior of the avalanche probability as well as $f^{-\phi}$ behavior for the power spectrum. However, our model displays nonuniversality: the exponents β and κ change continuously with λ. On the other hand the exponents which describe the fluctuations, ϕ and ζ, are universal, i.e. independent of λ. Kertész et al [8] have shown that a great class of sandpile automata yield f^{-2} behavior. Thus it is understandable why ϕ is constant. In contrast reasons why the roughness exponent is independent of the toppling parameter are not known to us. Generalization of this model to higher dimensions and an investigation of the behavior for very small and very large values of λ where power-law behavior apparently breaks down is under way.

References

[1] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987);
P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).

[2] L. P. Kadanoff, S. R. Nagel, L. Wu and S. M. Zhou, Phys.Rev. A **39**, 6524 (1988).

[3] J. Krug, J. E. S. Socolar and G. Grinstein, Phys. Rev. A **46**,4479 (1992).

[4] T. Hwa and M. Kardar, Phys. Rev. A **45**, 7002 (1992).

[5] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).

[6] K. Christensen and Z. Olami, Phys. Rev. A **46**, 1829 (1992).

[7] S. Lübeck, K. D. Usadel, (to be published).

[8] J. Kertész and L. B. Kiss, J. Phys. A **23**, 433 (1990).